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Fourth-order compact formulation of Navier–Stokes equations and driven cavity flow at high Reynolds numbers

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SUMMARY

A new fourth-order compact formulation for the steady 2-D incompressible Navier–Stokes equations is presented. The formulation is in the same form of the Navier–Stokes equations such that any numerical method that solve the Navier–Stokes equations can easily be applied to this fourth-order compact formulation. In particular, in this work the formulation is solved with an efficient numerical method that requires the solution of tridiagonal systems using a fine grid mesh of 601×601. Using this formulation, the steady 2-D incompressible flow in a driven cavity is solved up to Reynolds number with $Re = 20\,000$ fourth-order spatial accuracy. Detailed solutions are presented. Copyright © 2005 John Wiley & Sons, Ltd.

KEY WORDS: high-order compact scheme; HOC; steady 2-D incompressible N–S equations; driven cavity flow; high Reynolds number solutions

1. INTRODUCTION

High-order compact (HOC) formulations are becoming more popular in computational fluid dynamics (CFD) field of study. Compact formulations provide more accurate solutions in a compact stencil.

In finite differences, a standard three-point discretization provides second-order spatial accuracy and this type of discretization is very widely used. When a high-order spatial discretization is desired, i.e. fourth-order accuracy, then a five-point discretization have to be used. However, in a five point discretization there is a complexity in handling the points near the boundaries.

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High-order compact schemes provide fourth-order spatial accuracy in a 3×3 stencil and this type of compact formulations does not have the complexity near the boundaries that a standard wide (five-point) fourth-order formulation would have.

Dennis and Hudson [1], MacKinnon and Johnson [2], Gupta *et al.* [3], Spotz and Carey [4] and Li *et al.* [5] have demonstrated the efficiency of the HOC schemes on the streamfunction and vorticity formulation of 2-D steady incompressible Navier–Stokes equations.

In the literature, it is possible to find numerous different types of iterative numerical methods for the Navier–Stokes equations. These numerical methods, however, could not be easily used in HOC schemes because of the final form of the HOC formulations used in References [1–5]. This fact might be counted as a disadvantage of HOC formulations that the coding stage is rather complex due to the resulting stencil used in these studies. It would be very useful if any numerical method for the solution of Navier–Stokes equations described in books and papers could be easily applied to HOC formulations.

In this study, we will present a new fourth-order compact formulation. The difference of this formulation with References [1–5] is not in the way that the fourth-order compact scheme is obtained. The main difference, however, is in the way that the final form of the equations are written. The main advantage of this formulation is that, any iterative numerical method used for Navier–Stokes equations, can be easily applied to this new HOC formulation, since the final form of the presented HOC formulation is in the same form with the Navier–Stokes equations. Moreover, if someone already have a second-order accurate ($\mathcal{O}\Delta x^2$) code for the solution of steady 2-D incompressible Navier–Stokes equations, they can easily convert their existing code to fourth-order accuracy ($\mathcal{O}\Delta x^4$) by just adding some coefficients into their existing code. Using this new compact formulation, we have solved the steady 2-D incompressible driven cavity flow at very high Reynolds numbers using a very fine grid mesh to demonstrate the efficiency of this new formulation.

2. FOURTH-ORDER COMPACT FORMULATION

In non-dimensional form, steady 2-D incompressible Navier–Stokes equations in streamfunction (ψ) and vorticity (ω) formulation are given as

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\omega \tag{1}$$

$$\frac{1}{Re}\frac{\partial^2 \omega}{\partial x^2} + \frac{1}{Re}\frac{\partial^2 \omega}{\partial y^2} = \frac{\partial \psi}{\partial y}\frac{\partial \omega}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \omega}{\partial y} \tag{2}$$

where x and y are the Cartesian coordinates and Re is the Reynolds number. For first- and second-order derivatives the following discretizations are fourth-order accurate:

$$\frac{\partial \phi}{\partial x} = \phi_x - \frac{\Delta x^2}{6}\frac{\partial^3 \phi}{\partial x^3} + \mathcal{O}(\Delta x^4) \tag{3}$$

$$\frac{\partial^2 \phi}{\partial x^2} = \phi_{xx} - \frac{\Delta x^2}{12}\frac{\partial^4 \phi}{\partial x^4} + \mathcal{O}(\Delta x^4) \tag{4}$$

where ϕ_x and ϕ_{xx} are standard second-order central discretizations such that

$$\phi_x = \frac{\phi_{i+1} - \phi_{i-1}}{2\Delta x} \tag{5}$$

$$\phi_{xx} = \frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{\Delta x^2} \tag{6}$$

If we apply the discretizations in Equations (3) and (4) to Equations (1) and (2), we obtain the following equations:

$$\psi_{xx} + \psi_{yy} - \frac{\Delta x^2}{12}\frac{\partial^4\psi}{\partial x^4} - \frac{\Delta y^2}{12}\frac{\partial^4\psi}{\partial y^4} + \mathcal{O}(\Delta x^4, \Delta y^4) = -\omega \tag{7}$$

$$\frac{1}{Re}\omega_{xx} + \frac{1}{Re}\omega_{yy} - \frac{1}{Re}\frac{\Delta x^2}{12}\frac{\partial^4\omega}{\partial x^4} - \frac{1}{Re}\frac{\Delta y^2}{12}\frac{\partial^4\omega}{\partial y^4} + \mathcal{O}(\Delta x^4, \Delta y^4) = \psi_y\omega_x - \psi_x\omega_y$$

$$-\frac{\Delta y^2}{6}\omega_x\frac{\partial^3\psi}{\partial y^3} - \frac{\Delta x^2}{6}\psi_y\frac{\partial^3\omega}{\partial x^3} + \frac{\Delta x^2}{6}\omega_y\frac{\partial^3\psi}{\partial x^3} + \frac{\Delta y^2}{6}\psi_x\frac{\partial^3\omega}{\partial y^3} + \mathcal{O}(\Delta x^4, \Delta x^2, \Delta y^2, \Delta y^4) \tag{8}$$

In these equations we have third and fourth derivatives ($\partial^3/\partial x^3$, $\partial^3/\partial y^3$, $\partial^4/\partial x^4$ and $\partial^4/\partial y^4$) of streamfunction and vorticity (ψ and ω) variables. In order to find an expression for these derivatives we use Equations (1) and (2). For example, when we take the first and second x-derivative ($\partial/\partial x$ and $\partial^2/\partial x^2$) of the streamfunction equation (1) we obtain

$$\frac{\partial^3\psi}{\partial x^3} = -\frac{\partial\omega}{\partial x} - \frac{\partial^3\psi}{\partial x\partial y^2} \tag{9}$$

$$\frac{\partial^4\psi}{\partial x^4} = -\frac{\partial^2\omega}{\partial x^2} - \frac{\partial^4\psi}{\partial x^2\partial y^2} \tag{10}$$

And also, by taking the first and second y-derivative ($\partial/\partial y$ and $\partial^2/\partial y^2$) of the streamfunction equation (1) we obtain

$$\frac{\partial^3\psi}{\partial y^3} = -\frac{\partial\omega}{\partial y} - \frac{\partial^3\psi}{\partial x^2\partial y} \tag{11}$$

$$\frac{\partial^4\psi}{\partial y^4} = -\frac{\partial^2\omega}{\partial y^2} - \frac{\partial^4\psi}{\partial x^2\partial y^2} \tag{12}$$

Using standard second-order central discretizations given in Table I, these equations can be written as

$$\frac{\partial^3\psi}{\partial x^3} = -\omega_x - \psi_{xyy} + \mathcal{O}(\Delta x^2, \Delta y^2) \tag{13}$$

$$\frac{\partial^4\psi}{\partial x^4} = -\omega_{xx} - \psi_{xxyy} + \mathcal{O}(\Delta x^2, \Delta y^2) \tag{14}$$

Table I. Standard second-order central discretizations, $\mathcal{O}(\Delta x^2, \Delta y^2)$.

ϕ_x	$\dfrac{\phi_{i+1,j} - \phi_{i-1,j}}{2\Delta x}$
ϕ_y	$\dfrac{\phi_{i,j+1} - \phi_{i,j-1}}{2\Delta y}$
ϕ_{xx}	$\dfrac{\phi_{i+1,j} - 2\phi_{i,j} + \phi_{i-1,j}}{\Delta x^2}$
ϕ_{yy}	$\dfrac{\phi_{i,j+1} - 2\phi_{i,j} + \phi_{i,j-1}}{\Delta y^2}$
ϕ_{xy}	$\dfrac{\phi_{i+1,j+1} - \phi_{i-1,j+1} - \phi_{i+1,j-1} + \phi_{i-1,j-1}}{4\Delta x \Delta y}$
ϕ_{xxy}	$\dfrac{\phi_{i+1,j+1} - 2\phi_{i,j+1} + \phi_{i-1,j+1} - \phi_{i+1,j-1} + 2\phi_{i,j-1} - \phi_{i-1,j-1}}{2\Delta x^2 \Delta y}$
ϕ_{xyy}	$\dfrac{\phi_{i+1,j+1} - 2\phi_{i+1,j} + \phi_{i+1,j-1} - \phi_{i-1,j+1} - 2\phi_{i-1,j} + \phi_{i-1,j-1}}{2\Delta x \Delta y^2}$
ϕ_{xxyy}	$\dfrac{\phi_{i+1,j+1} - 2\phi_{i,j+1} + \phi_{i-1,j+1} - 2\phi_{i+1,j} + 4\phi_{i,j} - 2\phi_{i-1,j} + \phi_{i+1,j-1} - 2\phi_{i,j-1} + \phi_{i-1,j-1}}{\Delta x^2 \Delta y^2}$

$$\frac{\partial^3 \psi}{\partial y^3} = -\omega_y - \psi_{xxy} + \mathcal{O}(\Delta x^2, \Delta y^2) \tag{15}$$

$$\frac{\partial^4 \psi}{\partial y^4} = -\omega_{yy} - \psi_{xxyy} + \mathcal{O}(\Delta x^2, \Delta y^2) \tag{16}$$

When we substitute Equations (14) and (16) into Equation (7) we obtain the following finite difference equation:

$$\psi_{xx} + \psi_{yy} = -\omega - \frac{\Delta x^2}{12}\omega_{xx} - \frac{\Delta y^2}{12}\omega_{yy} - \left(\frac{\Delta x^2}{12} + \frac{\Delta y^2}{12}\right)\psi_{xxyy} + \mathcal{O}(\Delta x^4, \Delta x^2 \Delta y^2, \Delta y^4) \tag{17}$$

We note that the solution of Equation (17) is also a solution to streamfunction Equation (1) with fourth-order spatial accuracy. Therefore, if we numerically solve Equation (17), the solution we obtain will satisfy the streamfunction equation up to fourth-order accuracy.

In order to obtain a fourth-order approximation for the vorticity equation (2), we follow the same procedure. When we take the first and second derivatives of the vorticity equation (2) with respect to x- and y-coordinates we obtain

$$\frac{\partial^3 \omega}{\partial x^3} = Re\frac{\partial^2 \psi}{\partial x \partial y}\frac{\partial \omega}{\partial x} + Re\frac{\partial \psi}{\partial y}\frac{\partial^2 \omega}{\partial x^2} - Re\frac{\partial^2 \psi}{\partial x^2}\frac{\partial \omega}{\partial y} - Re\frac{\partial \psi}{\partial x}\frac{\partial^2 \omega}{\partial x \partial y} - \frac{\partial^3 \omega}{\partial x \partial y^2} \tag{18}$$

$$\frac{\partial^4\omega}{\partial x^4} = Re\frac{\partial^3\psi}{\partial x^2\partial y}\frac{\partial\omega}{\partial x} + Re\frac{\partial^2\psi}{\partial x\partial y}\frac{\partial^2\omega}{\partial x^2} + Re\frac{\partial^2\psi}{\partial x\partial y}\frac{\partial^2\omega}{\partial x^2} + Re\frac{\partial\psi}{\partial y}\frac{\partial^3\omega}{\partial x^3}$$

$$-Re\frac{\partial^3\psi}{\partial x^3}\frac{\partial\omega}{\partial y} - Re\frac{\partial^2\psi}{\partial x^2}\frac{\partial^2\omega}{\partial x\partial y} - Re\frac{\partial^2\psi}{\partial x^2}\frac{\partial^2\omega}{\partial x\partial y} - Re\frac{\partial\psi}{\partial x}\frac{\partial^3\omega}{\partial x^2\partial y} - \frac{\partial^4\omega}{\partial x^2\partial y^2} \tag{19}$$

$$\frac{\partial^3\omega}{\partial y^3} = Re\frac{\partial^2\psi}{\partial y^2}\frac{\partial\omega}{\partial x} + Re\frac{\partial\psi}{\partial y}\frac{\partial^2\omega}{\partial x\partial y} - Re\frac{\partial^2\psi}{\partial x\partial y}\frac{\partial\omega}{\partial y} - Re\frac{\partial\psi}{\partial x}\frac{\partial^2\omega}{\partial y^2} - \frac{\partial^3\omega}{\partial x^2\partial y} \tag{20}$$

$$\frac{\partial^4\omega}{\partial y^4} = Re\frac{\partial^3\psi}{\partial y^3}\frac{\partial\omega}{\partial x} + Re\frac{\partial^2\psi}{\partial y^2}\frac{\partial^2\omega}{\partial x\partial y} + Re\frac{\partial^2\psi}{\partial y^2}\frac{\partial^2\omega}{\partial x\partial y} + Re\frac{\partial\psi}{\partial y}\frac{\partial^3\omega}{\partial x\partial y^2}$$

$$-Re\frac{\partial^3\psi}{\partial x\partial y^2}\frac{\partial\omega}{\partial y} - Re\frac{\partial^2\psi}{\partial x\partial y}\frac{\partial^2\omega}{\partial y^2} - Re\frac{\partial^2\psi}{\partial x\partial y}\frac{\partial^2\omega}{\partial y^2} - Re\frac{\partial\psi}{\partial x}\frac{\partial^3\omega}{\partial y^3} - \frac{\partial^4\omega}{\partial x^2\partial y^2} \tag{21}$$

If we substitute Equations (18) and (20) for the third derivatives of vorticity ($\partial^3\omega/\partial x^3$ and $\partial^3\omega/\partial y^3$) into Equations (8), (19) and (21) and also if we substitute Equations (13) and (15) for the third derivatives of streamfunction ($\partial^3\psi/\partial x^3$ and $\partial^3\psi/\partial y^3$) into Equations (8), (19) and (21), and finally, if we substitute Equations (19) and (21) for the fourth derivatives of vorticity ($\partial^4\omega/\partial x^4$ and $\partial^4\omega/\partial y^4$) into Equation (8), then we obtain the following:

$$\omega_{xx} + \omega_{yy} - Re\frac{\Delta x^2}{6}\psi_{xy}\omega_{xx} + Re\frac{\Delta y^2}{6}\psi_{xy}\omega_{yy} + Re^2\frac{\Delta x^2}{12}\psi_y\psi_y\omega_{xx} + Re^2\frac{\Delta y^2}{12}\psi_x\psi_x\omega_{yy}$$

$$= Re\psi_y\omega_x - Re\psi_x\omega_y + Re\left(\frac{\Delta x^2}{12} + \frac{\Delta y^2}{12}\right)\psi_{xxy}\omega_x - Re\left(\frac{\Delta x^2}{12} + \frac{\Delta y^2}{12}\right)\psi_{xyy}\omega_y$$

$$-Re^2\frac{\Delta x^2}{12}\psi_y\psi_{xy}\omega_x + Re^2\frac{\Delta y^2}{12}\psi_x\psi_{yy}\omega_x + Re^2\frac{\Delta x^2}{12}\psi_y\psi_{xx}\omega_y - Re^2\frac{\Delta y^2}{12}\psi_x\psi_{xy}\omega_y$$

$$+Re\left(\frac{\Delta x^2}{12} + \frac{\Delta y^2}{12}\right)\psi_y\omega_{xyy} - Re\left(\frac{\Delta x^2}{12} + \frac{\Delta y^2}{12}\right)\psi_x\omega_{xxy} - Re\frac{\Delta x^2}{6}\psi_{xx}\omega_{xy}$$

$$+Re\frac{\Delta y^2}{6}\psi_{yy}\omega_{xy} + Re^2\left(\frac{\Delta x^2}{12} + \frac{\Delta y^2}{12}\right)\psi_x\psi_y\omega_{xy} - Re\left(\frac{\Delta x^2}{12} - \frac{\Delta y^2}{12}\right)\omega_x\omega_y$$

$$-\left(\frac{\Delta x^2}{12} + \frac{\Delta y^2}{12}\right)\omega_{xxyy} + \mathcal{O}(\Delta x^4, \Delta x^2\Delta y^2, \Delta y^4) \tag{22}$$

Again we note that the solution of Equation (22) satisfy the vorticity equation (2) with fourth-order accuracy.

As the final form of our HOC scheme, we prefer to write Equations (17) and (22) as

$$\psi_{xx} + \psi_{yy} = -\omega + A \tag{23}$$

$$\frac{1}{Re}(1+B)\omega_{xx}+\frac{1}{Re}(1+C)\omega_{yy}=(\psi_y+D)\omega_x-(\psi_x+E)\omega_y+F \tag{24}$$

where

$$\begin{aligned}
A &= -\frac{\Delta x^2}{12}\omega_{xx}-\frac{\Delta y^2}{12}\omega_{yy}-\left(\frac{\Delta x^2}{12}+\frac{\Delta y^2}{12}\right)\psi_{xxyy}\\
B &= -Re\frac{\Delta x^2}{6}\psi_{xy}+Re^2\frac{\Delta x^2}{12}\psi_y\psi_y\\
C &= Re\frac{\Delta y^2}{6}\psi_{xy}+Re^2\frac{\Delta y^2}{12}\psi_x\psi_x\\
D &= \left(\frac{\Delta x^2}{12}+\frac{\Delta y^2}{12}\right)\psi_{xxy}-Re\frac{\Delta x^2}{12}\psi_y\psi_{xy}+Re\frac{\Delta y^2}{12}\psi_x\psi_{yy}\\
E &= \left(\frac{\Delta x^2}{12}+\frac{\Delta y^2}{12}\right)\psi_{xyy}-Re\frac{\Delta x^2}{12}\psi_y\psi_{xx}+Re\frac{\Delta y^2}{12}\psi_x\psi_{xy}\\
F &= \left(\frac{\Delta x^2}{12}+\frac{\Delta y^2}{12}\right)\psi_y\omega_{xyy}-\left(\frac{\Delta x^2}{12}+\frac{\Delta y^2}{12}\right)\psi_x\omega_{xxy}-\frac{\Delta x^2}{6}\psi_{xx}\omega_{xy}\\
&\quad+\frac{\Delta y^2}{6}\psi_{yy}\omega_{xy}+Re\left(\frac{\Delta x^2}{12}+\frac{\Delta y^2}{12}\right)\psi_x\psi_y\omega_{xy}-\left(\frac{\Delta x^2}{12}-\frac{\Delta y^2}{12}\right)\omega_x\omega_y\\
&\quad-\frac{1}{Re}\left(\frac{\Delta x^2}{12}+\frac{\Delta y^2}{12}\right)\omega_{xxyy}
\end{aligned} \tag{25}$$

We note that the finite difference equations (23) and (24) are fourth-order accurate ($\mathcal{O}(\Delta x^4, \Delta x^2\Delta y^2, \Delta y^4)$) approximation of the streamfunction and vorticity equations (1) and (2). In Equations (23) and (24), however, if A, B, C, D, E and F are chosen to be equal to 0 then the finite difference equations (23) and (24) simply become

$$\psi_{xx}+\psi_{yy}=-\omega \tag{26}$$

$$\frac{1}{Re}\omega_{xx}+\frac{1}{Re}\omega_{yy}=\psi_y\omega_x-\psi_x\omega_y \tag{27}$$

Equations (26) and (27) are the standard second-order accurate ($\mathcal{O}(\Delta x^2, \Delta y^2)$) approximation of the streamfunction and vorticity equations (1) and (2). When we use Equations (23) and (24) for the numerical solution of 2-D steady incompressible Navier–Stokes equations, we can easily switch between second and fourth-order accuracy just by using homogeneous values for the coefficients A, B, C, D, E and F or by using the expressions defined in Equation (25) in the code.

In Equations (23)–(25) instead of finite difference discretizations if we substitute for partial derivatives we obtain the following differential equations:

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=-\omega+A \tag{28}$$

$$\frac{1}{Re}(1+B)\frac{\partial^2\omega}{\partial x^2}+\frac{1}{Re}(1+C)\frac{\partial^2\omega}{\partial y^2}=\left(\frac{\partial\psi}{\partial y}+D\right)\frac{\partial\omega}{\partial x}-\left(\frac{\partial\psi}{\partial x}+E\right)\frac{\partial\omega}{\partial y}+F \tag{29}$$

$$A=-\frac{\Delta x^2}{12}\frac{\partial^2\omega}{\partial x^2}-\frac{\Delta y^2}{12}\frac{\partial^2\omega}{\partial y^2}-\left(\frac{\Delta x^2}{12}+\frac{\Delta y^2}{12}\right)\frac{\partial^4\psi}{\partial x^2\partial y^2}$$

$$B=-Re\frac{\Delta x^2}{6}\frac{\partial^2\psi}{\partial x\partial y}+Re^2\frac{\Delta x^2}{12}\frac{\partial\psi}{\partial y}\frac{\partial\psi}{\partial y}$$

$$C=Re\frac{\Delta y^2}{6}\frac{\partial^2\psi}{\partial x\partial y}+Re^2\frac{\Delta y^2}{12}\frac{\partial\psi}{\partial x}\frac{\partial\psi}{\partial x}$$

$$D=\left(\frac{\Delta x^2}{12}+\frac{\Delta y^2}{12}\right)\frac{\partial^3\psi}{\partial x^2\partial y}-Re\frac{\Delta x^2}{12}\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y}+Re\frac{\Delta y^2}{12}\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}$$

$$E=\left(\frac{\Delta x^2}{12}+\frac{\Delta y^2}{12}\right)\frac{\partial^3\psi}{\partial x\partial y^2}-Re\frac{\Delta x^2}{12}\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x^2}+Re\frac{\Delta y^2}{12}\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial x\partial y}$$

$$\begin{aligned}F=&\left(\frac{\Delta x^2}{12}+\frac{\Delta y^2}{12}\right)\frac{\partial\psi}{\partial y}\frac{\partial^3\omega}{\partial x\partial y^2}-\left(\frac{\Delta x^2}{12}+\frac{\Delta y^2}{12}\right)\frac{\partial\psi}{\partial x}\frac{\partial^3\omega}{\partial x^2\partial y}-\frac{\Delta x^2}{6}\frac{\partial^2\psi}{\partial x^2}\frac{\partial^2\omega}{\partial x\partial y}\\&+\frac{\Delta y^2}{6}\frac{\partial^2\psi}{\partial y^2}\frac{\partial^2\omega}{\partial x\partial y}+Re\left(\frac{\Delta x^2}{12}+\frac{\Delta y^2}{12}\right)\frac{\partial\psi}{\partial x}\frac{\partial\psi}{\partial y}\frac{\partial^2\omega}{\partial x\partial y}-\left(\frac{\Delta x^2}{12}-\frac{\Delta y^2}{12}\right)\frac{\partial\omega}{\partial x}\frac{\partial\omega}{\partial y}\\&-\frac{1}{Re}\left(\frac{\Delta x^2}{12}+\frac{\Delta y^2}{12}\right)\frac{\partial^4\omega}{\partial x^2\partial y^2}\end{aligned} \tag{30}$$

We note that the numerical solutions of Equations (28) and (29), strictly provided that second-order discretizations in Table I are used and also strictly provided that a uniform grid mesh with Δx and Δy is used, are fourth-order accurate to streamfunction and vorticity equations (1) and (2). We prefer to call Equations (28) and (29) Fourth-Order Navier–Stokes (FONS) equations. The only difference between FONS equations (28) and (29) and Navier–Stokes (NS) equations (1) and (2) are the coefficients A, B, C, D, E and F. In fact the NS equations are a subset of the FONS equations. We note that FONS equations (28) and (29) are in the same form with Navier–Stokes equations (1) and (2), therefore, any iterative numerical method (such as SOR, ADI, factorization schemes, pseudo time iterations, etc.) used to solve streamfunction and vorticity equations (1) and (2) can also be easily applied to fourth-order equations (28) and (29). Moreover, any existing code that solve the streamfunction and vorticity equations with second-order accuracy can easily be modified to provide fourth-order accuracy just by adding the coefficients A, B, C, D, E and F into the existing code to obtain the solution of FONS equations. Of course, when the coefficients A, B, C, D, E and F are added into a second-order accurate code to obtain fourth-order accuracy, evaluating these coefficients would require extra CPU work. This might be considered as the cost of increasing accuracy from second- to fourth-order.

3. NUMERICAL METHOD

Recently Erturk *et al.* [6] have presented a new, stable and efficient numerical method that solve the streamfunction and vorticity equations. The numerical method solve the governing steady equations through iterations in the pseudo time. In this study, we will apply the numerical method Erturk *et al.* [6] have proposed, to FONS equations (28) and (29) and solve the steady driven cavity flow with fourth-order accuracy. For details about the numerical method, the reader is referred to Erturk *et al.* [6]. When we apply the numerical method to Equations (28) and (29) the equations take the following form:

$$\left(1-\Delta t\frac{\partial^2}{\partial x^2}\right)\left(1-\Delta t\frac{\partial^2}{\partial y^2}\right)\psi^{n+1}=\psi^n+\Delta t\omega^n-\Delta t A^n+\left(\Delta t\frac{\partial^2}{\partial x^2}\right)\left(\Delta t\frac{\partial^2}{\partial y^2}\right)\psi^n \quad (31)$$

$$\begin{aligned}&\left(1-\Delta t(1+B^n)\frac{1}{Re}\frac{\partial^2}{\partial x^2}+\Delta t\left(\frac{\partial\psi}{\partial y}+D\right)^n\frac{\partial}{\partial x}\right)\\ &\quad\times\left(1-\Delta t(1+C^n)\frac{1}{Re}\frac{\partial^2}{\partial y^2}-\Delta t\left(\frac{\partial\psi}{\partial x}+E\right)^n\frac{\partial}{\partial y}\right)\omega^{n+1}\\ &\quad=\omega^n-\Delta t F^n+\left(\Delta t(1+B^n)\frac{1}{Re}\frac{\partial^2}{\partial x^2}-\Delta t\left(\frac{\partial\psi}{\partial y}+D\right)^n\frac{\partial}{\partial x}\right)\\ &\qquad\times\left(\Delta t(1+C^n)\frac{1}{Re}\frac{\partial^2}{\partial y^2}+\Delta t\left(\frac{\partial\psi}{\partial x}+E\right)^n\frac{\partial}{\partial y}\right)\omega^n\end{aligned} \quad (32)$$

The solution methodology of these two equations are quite simple. First the streamfunction equation (31) is solved in two steps. For streamfunction equation, a new variable f is defined as

$$\left(1-\Delta t\frac{\partial^2}{\partial y^2}\right)\psi^{n+1}=f \quad (33)$$

Using this variable in Equation (31) we obtain

$$\left(1-\Delta t\frac{\partial^2}{\partial x^2}\right)f=\psi^n+\Delta t\omega^n-\Delta t A^n+\left(\Delta t\frac{\partial^2}{\partial x^2}\right)\left(\Delta t\frac{\partial^2}{\partial y^2}\right)\psi^n \quad (34)$$

In this equation, the only unknown is the variable f. We first solve this equation for f by solving a tridiagonal system. After this, when we obtain the value of f at every grid point we solve Equation (33) for streamfunction (ψ^{n+1}) by solving another tridiagonal system.

After solving the streamfunction equation (31), we solve the vorticity equation (32). For this, similarly, we introduce a new variable g which is defined as

$$\left(1-\Delta t(1+C^n)\frac{1}{Re}\frac{\partial^2}{\partial y^2}-\Delta t\left(\frac{\partial\psi}{\partial x}+E\right)^n\frac{\partial}{\partial y}\right)\omega^{n+1}=g \quad (35)$$

Using this variable in Equation (32), we obtain

$$\begin{aligned}
&\left(1-\Delta t(1+B^n)\frac{1}{Re}\frac{\partial^2}{\partial x^2}+\Delta t\left(\frac{\partial\psi}{\partial y}+D\right)^n\frac{\partial}{\partial x}\right)g\\
&=\omega^n-\Delta tF^n\\
&\quad+\left(\Delta t(1+B^n)\frac{1}{Re}\frac{\partial^2}{\partial x^2}-\Delta t\left(\frac{\partial\psi}{\partial y}+D\right)^n\frac{\partial}{\partial x}\right)\\
&\quad\times\left(\Delta t(1+C^n)\frac{1}{Re}\frac{\partial^2}{\partial y^2}+\Delta t\left(\frac{\partial\psi}{\partial x}+E\right)^n\frac{\partial}{\partial y}\right)\omega^n
\end{aligned} \tag{36}$$

In this equation the only unknown is the variable g. By solving a tridiagonal system, we obtain the value of g at every grid point. Then we solve Equation (35) for vorticity (ω^{n+1}) by solving another tridiagonal system.

In a compact formulation, the stencil have 3×3 points. The solution at the first diagonal grid points near the corners of the cavity would require the vorticity values at the corner points. However, the corner points are singular points for vorticity. Gupta *et al.* [7] have introduced an explicit asymptotic solution in the neighbourhood of sharp corners. Similarly, Störtkuhl *et al.* [8] have presented an analytical asymptotic solutions near the corners of cavity and using finite element bilinear shape functions they also have presented a singularity removed boundary condition for vorticity at the corner points as well as at the wall points. We follow Störtkuhl *et al.* [8] and use the following expression for calculating vorticity values at the wall:

$$\frac{1}{3\Delta h^2}\begin{bmatrix}\bullet & \bullet & \bullet\\ \frac{1}{2} & -4 & \frac{1}{2}\\ 1 & 1 & 1\end{bmatrix}\psi+\frac{1}{9}\begin{bmatrix}\bullet & \bullet & \bullet\\ \frac{1}{2} & 2 & \frac{1}{2}\\ \frac{1}{4} & 1 & \frac{1}{2}\end{bmatrix}\omega=-\frac{V}{h} \tag{37}$$

where V is the speed of the wall which is equal to 1 for the moving top wall and equal to 0 for the three stationary walls. For corner points, we use the following expression for calculating the vorticity values:

$$\frac{1}{3\Delta h^2}\begin{bmatrix}\bullet & \bullet & \bullet\\ \bullet & -2 & \frac{1}{2}\\ \bullet & \frac{1}{2} & 1\end{bmatrix}\psi+\frac{1}{9}\begin{bmatrix}\bullet & \bullet & \bullet\\ \bullet & 1 & \frac{1}{2}\\ \bullet & \frac{1}{2} & \frac{1}{4}\end{bmatrix}\omega=-\frac{V}{2h} \tag{38}$$

where again V is equal to 1 for the upper two corners and it is equal to 0 for the bottom two corners. The reader is referred to Störtkuhl *et al.* [8] for details.

4. RESULTS AND DISCUSSIONS

The schematics of the driven cavity flow is given in Figure 1. In this figure the abbreviations BR, BL and TL refer to bottom-right, bottom-left and top-left corners of the cavity,

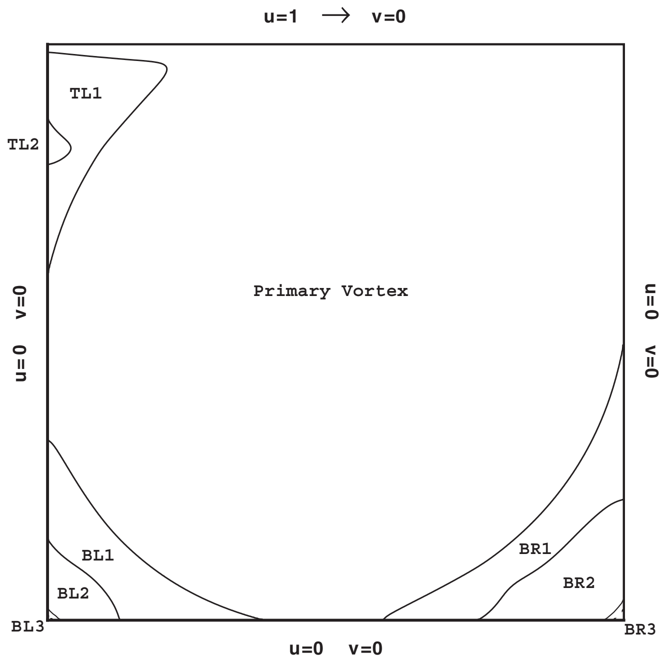

Figure 1. Schematic view of driven cavity flow.

respectively. The number following these abbreviations refer to the vortices that appear in the flow, which are numbered according to size.

For every Reynolds number considered, we have continued our iterations until, in the computational domain both the maximum residual of Equations (23) and (24), which are given as

$$R_\psi = \max(\text{abs}(|\psi_{xx}^{n+1} + \psi_{yy}^{n+1} + \omega^{n+1} - A^{n+1}|_{i,j})) \tag{39}$$

$$R_\omega = \max\left(\text{abs}\left(\left| \frac{1}{Re}(1 + B^{n+1})\omega_{xx}^{n+1} + \frac{1}{Re}(1 + C^{n+1})\omega_{yy}^{n+1} - (\psi_y^{n+1} + D^{n+1})\omega_x^{n+1} + (\psi_x^{n+1} + E^{n+1})\omega_y^{n+1} - F^{n+1} \right|_{i,j}\right)\right) \tag{40}$$

are less than 10^{-10}. Such a low value is chosen to ensure the accuracy of the solution. At these residual levels, the maximum absolute change in streamfunction value between two time steps, $(\max(|\psi^{n+1} - \psi^n|))$, was in the order of 10^{-16} and for vorticity, $(\max(|\omega^{n+1} - \omega^n|))$, it was in the order of 10^{-14}. Obviously, these convergence levels are far more less than satisfactory, however, such low values demonstrate the efficiency of the numerical method used in this study which was presented by Erturk *et al.* [6].

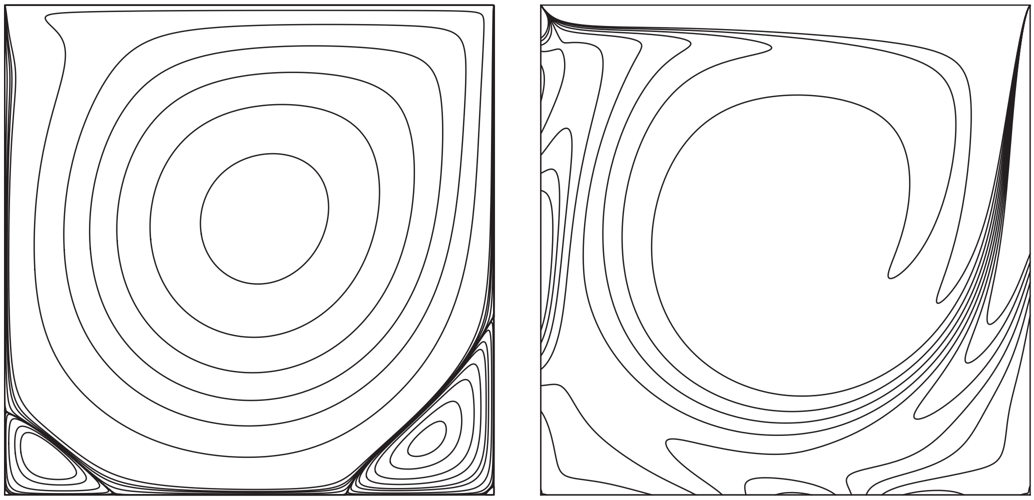

Figure 2. Streamfunction and vorticity contours for $Re = 1000$.

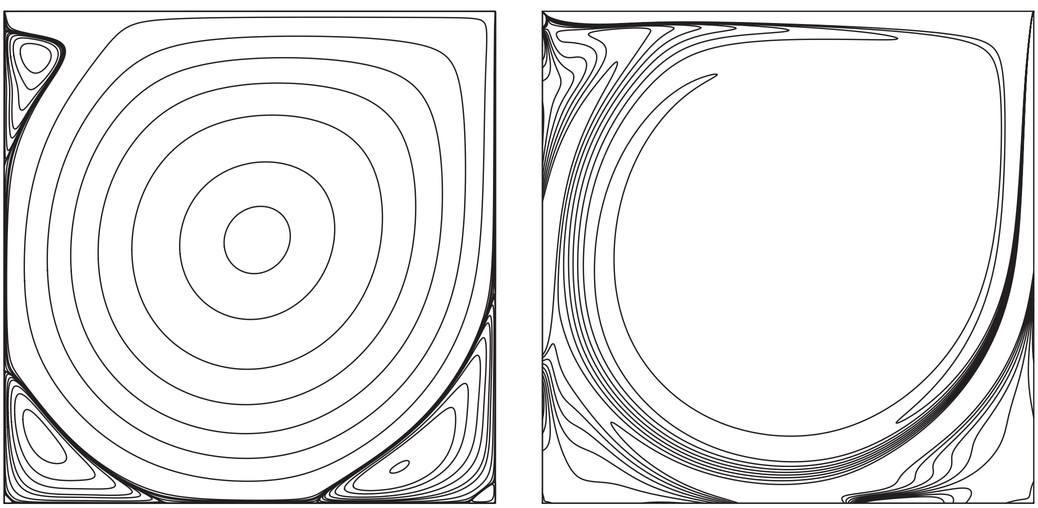

Figure 3. Streamfunction and vorticity contours for $Re = 5000$.

Using an efficient numerical method, Erturk *et al.* [6] have clearly shown that numerical solutions of driven cavity flow is computable for $Re > 10\,000$ when a grid mesh larger than 256×256 is used. With a grid mesh of 601×601 Erturk *et al.* [6] have solved the cavity flow up to $Re = 21\,000$ using the numerical method also used in this study. In order to be able to obtain solutions at high Reynolds numbers, following Erturk *et al.* [6], in this study we have used a large grid mesh with 601×601 grids. With this many number of grid points we obtained steady solutions of the cavity flow up to $Re = 20\,000$ with fourth-order accuracy.

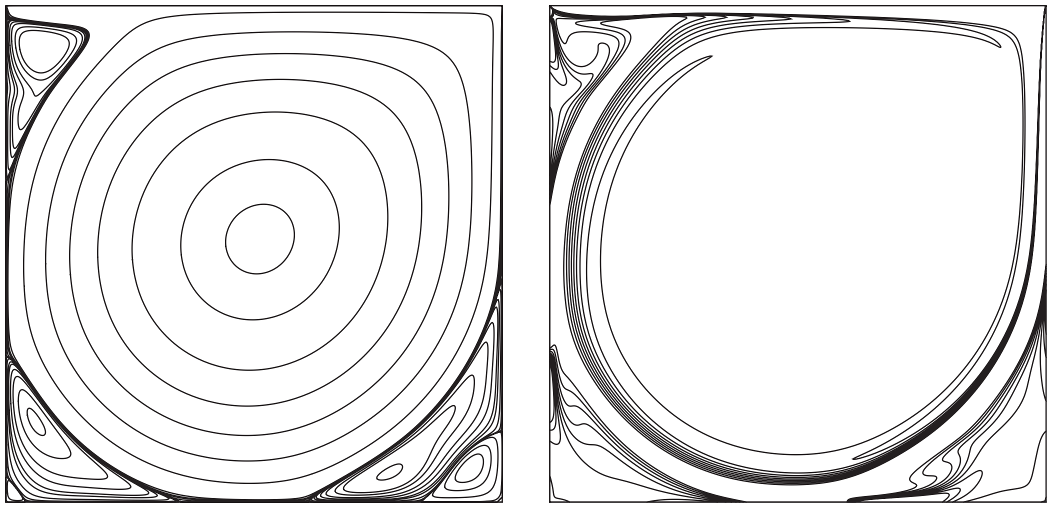

Figure 4. Streamfunction and vorticity contours for $Re = 10\,000$.

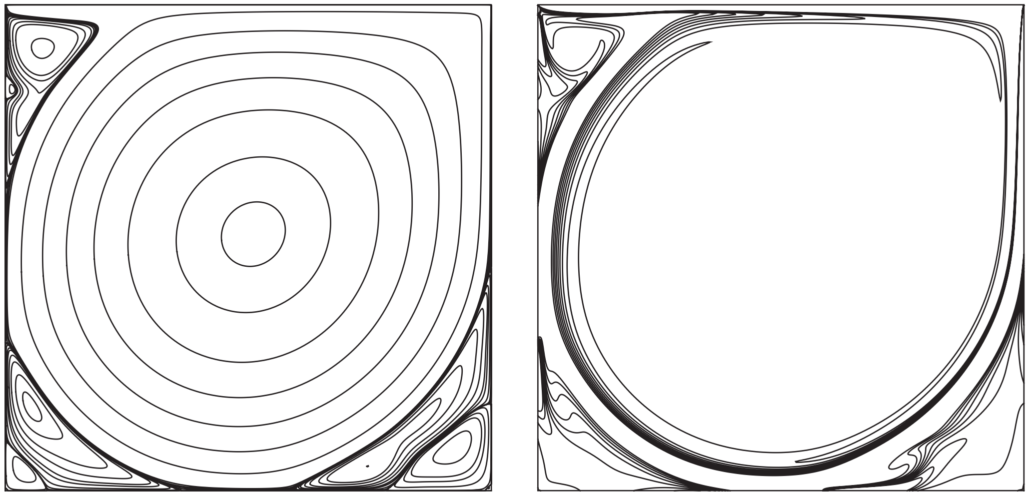

Figure 5. Streamfunction and vorticity contours for $Re = 15\,000$.

Figures 2–6 show the streamfunction and vorticity contours of the driven cavity flow between $Re = 1000$ and $Re = 20\,000$. These figures show the vortices that are formed in the flow field as the Reynolds number increases. From these contour figures, we conclude that the fourth-order compact formulation provides very smooth solutions.

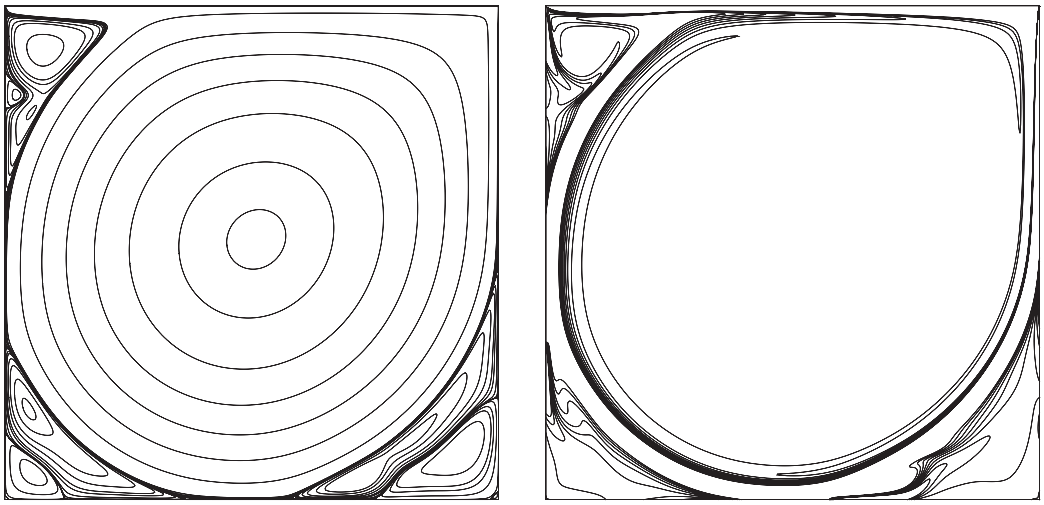

Figure 6. Streamfunction and vorticity contours for $Re = 20\,000$.

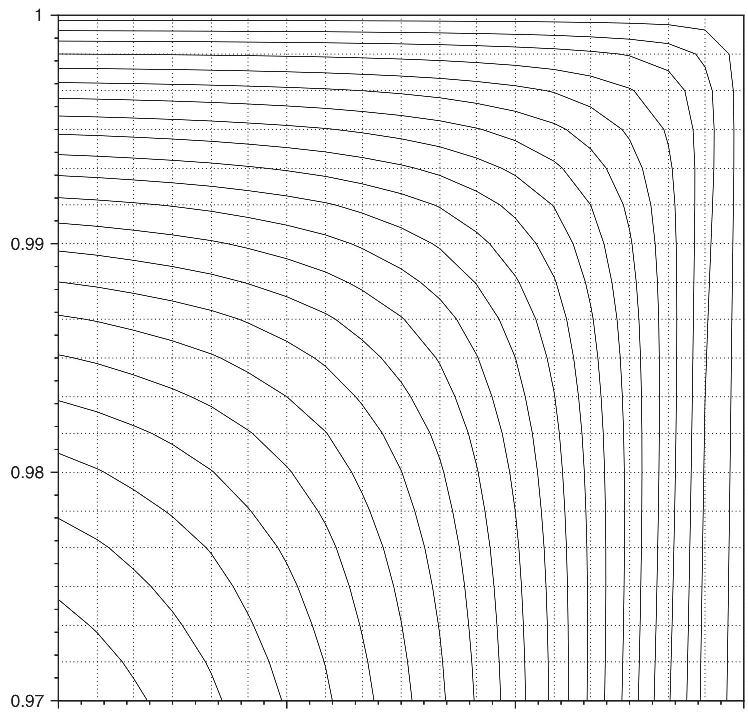

Figure 7. Streamfunction contours for $Re = 20\,000$, enlarged view of top-right corner.

Table II. Properties of primary and secondary vortices; streamfunction and vorticity values, (x, y) locations.

Re		1000	2500	5000	7500	10 000	12 500	15 000	17 500	20 000
Primary vortex	ψ	−0.118938	−0.121472	−0.122216	−0.122344	−0.122306	−0.122201	−0.122060	−0.121889	−0.121694
	ω	−2.067760	−1.976132	−1.940547	−1.926478	−1.918187	−1.912307	−1.907651	−1.903659	−1.900032
	(x, y)	(0.5300,0.5650)	(0.5200,0.5433)	(0.5150,0.5350)	(0.5133,0.5317)	(0.5117,0.5300)	(0.5117,0.5283)	(0.5100,0.5283)	(0.5100,0.5283)	(0.5100,0.5267)
BR1	ψ	0.17297E-02	0.26623E-02	0.30735E-02	0.32265E-02	0.31896E-02	0.30972E-02	0.30022E-02	0.29021E-02	0.28012E-02
	ω	1.118222	1.955594	2.739071	3.243925	3.756425	4.357323	4.965304	5.568522	6.125275
	(x, y)	(0.8633,0.1117)	(0.8333,0.0900)	(0.8050,0.0733)	(0.7900,0.0650)	(0.7750,0.0600)	(0.7600,0.0550)	(0.7450,0.0500)	(0.7333,0.0467)	(0.7200,0.0433)
BL1	ψ	0.23345E-03	0.93093E-03	0.13758E-02	0.15337E-02	0.16135E-02	0.16568E-02	0.16663E-02	0.16450E-02	0.16083E-02
	ω	0.354271	0.979831	1.514292	1.858636	2.182043	2.358513	2.490168	2.704118	2.885054
	(x, y)	(0.0833,0.0783)	(0.0833,0.1117)	(0.0733,0.1367)	(0.0650,0.1517)	(0.0583,0.1633)	(0.0550,0.1683)	(0.0533,0.1717)	(0.0500,0.1767)	(0.0483,0.1817)
BR2	ψ	−0.49242E-07	−0.12035E-06	−0.14271E-05	−0.32742E-04	−0.14014E-03	−0.25498E-03	−0.34006E-03	−0.40492E-03	−0.46117E-03
	ω	−0.76887E-02	−0.11935E-01	−0.33647E-01	−0.155445	−0.309618	−0.398536	−0.456877	−0.512862	−0.554507
	(x, y)	(0.9917,0.0067)	(0.9900,0.0083)	(0.9783,0.0183)	(0.9517,0.0417)	(0.9350,0.0683)	(0.9283,0.0817)	(0.9267,0.0883)	(0.9283,0.0967)	(0.9300,0.1050)
BL2	ψ	−0.65156E-08	−0.26763E-07	−0.65524E-07	−0.20189E-06	−0.11017E-05	−0.64493E-05	−0.22287E-04	−0.48296E-04	−0.78077E-04
	ω	−0.29579E-02	−0.93547E-02	−0.12183E-01	−0.16587E-01	−0.30153E-01	−0.74310E-01	−0.140685	−0.200227	−0.243259
	(x, y)	(0.0050,0.0050)	(0.0067,0.0067)	(0.0083,0.0083)	(0.0117,0.0117)	(0.0167,0.0200)	(0.0267,0.0317)	(0.0383,0.0417)	(0.0500,0.0483)	(0.0583,0.0533)
TL1	ψ	—	0.34284E-03	0.14459E-02	0.21311E-02	0.26237E-02	0.29949E-02	0.32865E-02	0.35245E-02	0.37247E-02
	ω	—	1.344040	2.115544	2.236986	2.322310	2.375671	2.424136	2.463058	2.497615
	(x, y)	—	(0.0433,0.8900)	(0.0633,0.9100)	(0.0667,0.9117)	(0.0700,0.9100)	(0.0733,0.9100)	(0.0767,0.9100)	(0.0783,0.9117)	(0.0800,0.9117)
BR3	ψ	—	—	0.42053E-10	0.75831E-09	0.37451E-08	0.78212E-08	0.11742E-07	0.16951E-07	0.27181E-07
	ω	—	—	0.66735E-03	0.70882E-03	0.24926E-02	0.37843E-02	0.33420E-02	0.42443E-02	0.49718E-02
	(x, y)	—	—	(0.9983,0.0017)	(0.9967,0.0017)	(0.9967,0.0050)	(0.9950,0.0050)	(0.9950,0.0050)	(0.9950,0.0067)	(0.9933,0.0067)
BL3	ψ	—	—	—	—	—	0.24668E-09	0.57300E-09	0.13069E-08	0.23598E-08
	ω	—	—	—	—	—	0.50277E-03	0.10067E-02	0.21872E-02	0.18079E-02
	(x, y)	—	—	—	—	—	(0.0017,0.0017)	(0.0017,0.0033)	(0.0033,0.0033)	(0.0033,0.0033)
TL2	ψ	—	—	—	—	—	−0.16253E-05	−0.15848E-04	−0.41183E-04	−0.70762E-04
	ω	—	—	—	—	—	−0.204044	−0.524342	−0.737721	−0.982968
	(x, y)	—	—	—	—	—	(0.0067,0.8300)	(0.0150,0.8250)	(0.0200,0.8217)	(0.0250,0.8200)

Table III. Minimum streamfunction values at the primary vortex for various Reynolds numbers.

Re	Present $\psi_{\min}$ ($\mathcal{O}\Delta x^4$)	Erturk *et al.* [6] $\psi_{\min}$ ($\mathcal{O}\Delta x^4$)
1000	−0.118938	−0.118939
2500	−0.121472	−0.121469
5000	−0.122216	−0.122213
7500	−0.122344	−0.122341
10 000	−0.122306	−0.122313
12 500	−0.122201	−0.122229
15 000	−0.122060	−0.122124
17 500	−0.121889	−0.122016
20 000	−0.121694	−0.121901

Table IV. Vorticity values at the centre of the primary vortex for various Reynolds numbers.

Re	Present ω ($\mathcal{O}\Delta x^4$)	Erturk *et al.* [6] ω ($\mathcal{O}\Delta x^4$)
1000	−2.067760	−2.067579
2500	−1.976132	−1.976096
5000	−1.940547	−1.940451
7500	−1.926478	−1.926282
10 000	−1.918187	−1.917919
12 500	−1.912307	−1.912072
15 000	−1.907651	−1.907602
17 500	−1.903659	−1.903975
20 000	−1.900032	−1.900891

In Figure 7 we plot a very enlarged view of the top-right corner (where the moving wall moves towards the stationary wall) of the streamfunction contour plot for the highest Reynolds number considered, $Re = 20\,000$. In this figure the dotted lines show the grid lines. As it is seen in this enlarged figure, fourth-order streamfunction contours are very smooth even at the first set of grid points near the corners.

Table II tabulates the streamfunction and vorticity values at the centre of the primary and secondary vortices and also the location of the centre of these vortices for future references. This table is in good agreement with that of Erturk *et al.* [6].

Using Richardson extrapolation on the solutions obtained with different grid meshes, Erturk *et al.* [6] have presented theoretically fourth- and sixth-order accurate ($\mathcal{O}\Delta x^4$ and $\mathcal{O}\Delta x^6$) streamfunction and vorticity values at the centre of the primary vortex. Tables III and IV compare the fourth-order compact scheme solutions of the streamfunction and the vorticity values at the centre of the primary vortex with the fourth-order ($\mathcal{O}\Delta x^4$) Richardson extrapolated solutions tabulated in Erturk *et al.* [6]. The present solutions and the solutions of Erturk *et al.* [6] agree with each other.

5. CONCLUSIONS

In this study a new fourth-order compact formulation is presented. The uniqueness of this formulation is that the final form of the HOC formulation is in the same form of the Navier–Stokes equations such that any numerical method that solve the Navier–Stokes equations can be easily applied to the FONS equations in order to obtain fourth-order accurate solutions ($\mathcal{O}\Delta x^4$). Moreover, with this formulation, any existing code that solve the Navier–Stokes equations with second-order accuracy ($\mathcal{O}\Delta x^2$) can be altered to provide fourth-order accurate ($\mathcal{O}\Delta x^4$) solutions just by adding some coefficients into the code at the expense of extra CPU work of evaluating these coefficients.

In this study, the presented fourth-order compact formulation is solved with a very efficient numerical method introduced by Erturk *et al.* [6]. Using a fine grid mesh of 601×601, as it was suggested by Erturk *et al.* [6] in order to be able to compute for high Reynolds numbers, the driven cavity flow is solved up to Reynolds number $Re = 20\,000$. The solutions obtained agree well with previous studies. The presented fourth-order accurate compact formulation is proved to be very efficient.

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